Technical Comments

Comments on "Approximate Theory for Terminal Velocity of a Freely Falling Body"

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Nomenclature

a = sound speed, fps
C_D = drag coefficient
g = acceleration due to gravity, ft/sec²
h = altitude, ft
m = mass, slugs
S = reference area, ft
V = velocity, fps; $\dot{V} = dV/dt$, ft/sec²
β = atmospheric density scale height inverse, ft^{-1}
ρ = atmospheric density, slugs/ft³

Subscripts

e = equilibrium
0 = value at zero Mach number
t = terminal

STILLEY[1] has linearized the vertical equation of motion of a freely falling body to find its terminal velocity. His result is similar to one derived by an alternate approach and previously published in this journal.[2] The latter method is amenable to improved approximation of the terminal velocity of a body with constant drag coefficient; however, neglect of the drag coefficient's Mach number variation limits the value of either result for bodies with terminal velocities in the transonic range.

The vertical equation of motion is

$$\dot{V} = g - C_D S \rho V^2/2m \tag{1}$$

Assuming that $\dot{V} = 0$, the equilibrium fall velocity is

$$V_e = (2mg/C_D S\rho)^{1/2} \tag{2}$$

as noted by Stilley. Because density is changing as altitude decreases, V also decreases, and $\dot{V} \neq 0$. Only when $\dot{V}$ is small is Eq. (2) a good approximation to fall velocity.

The next order of approximation is that $\dot{V}$ = constant $\neq 0$. Then $\ddot{V} = 0$ and an additional equation can be found by differentiating Eq. (1). This equation indicates that

$$\dot{V} = -\beta V^2/2 \tag{3}$$

where β is the scale height inverse of an exponential atmosphere and is equal to minus the density ratio gradient used by Stilley.

Substituting Eq. (3) in Eq. (1), the terminal velocity is found to be

$$V_t = \left(\frac{2mg}{C_D S\rho - m\beta}\right)^{1/2} = V_e\left(\frac{1}{1 - (m\beta/C_D S\rho)}\right)^{1/2} \tag{4}$$

If this result is expanded in a binomial series and all terms beyond first order are dropped,

$$V_t \simeq V_e\left(\frac{1}{1 - (m\beta/2C_D S\rho)}\right) \tag{5}$$

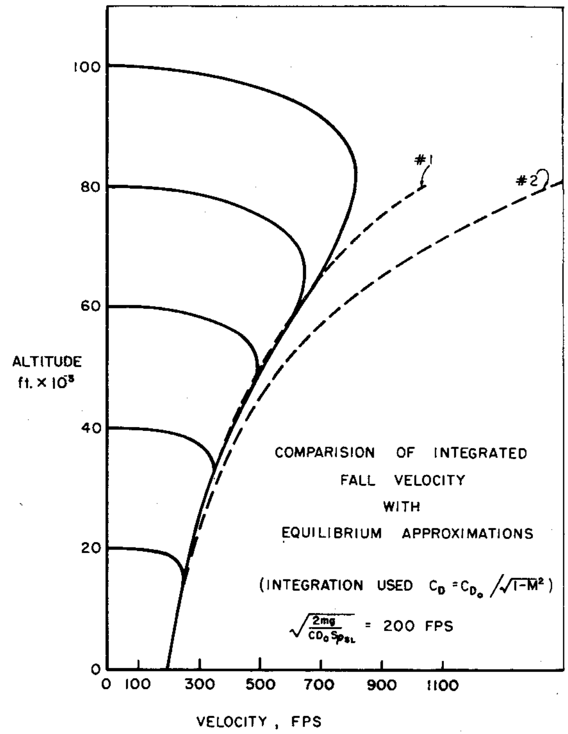

Fig. 1 Comparison of integrated fall velocity with equilibrium approximations. #1 = V_e, #2 = V_t; solid line indicates integrated result.

This is Stilley's result in a different notation. By repeated differentiation of Eq. (1), higher-order, closed-form solutions to the terminal velocity can be obtained. The nth-order approximation requires that the $(n + 1)$st derivative of V be set equal to zero.

It is dubious that higher-order approximations are of much value if the terminal velocity lies in the transonic region, since it is assumed that C_D is constant. It appears more logical to concentrate first on the effects of C_D variation. A functional relation between C_D, altitude, and velocity, e.g.,

$$C_D = C_{D_0}/\{1 - [V^2/a(h)^2]\}^{1/2} \tag{6}$$

for subsonic fall, must be defined and included in Eq. (1). Differentiation of Eq. (1) then yields terms with nonintegral powers of $(1 - V^2/a^2)$. A closed-form expression for V_t requires series expansion of these terms, as well as solution of a higher-order algebraic equation.

Before carrying out such a solution, it is instructive to examine a comparison of V_e, V_t, and an integrated velocity profile in Fig. 1. The integrated profile uses Eq. (6) for its drag variation. Here, V_e is a better approximation to actual terminal velocity than V_t once the initial transient has died out. The increased C_D effectively cancels the density gradient effect in this case. Nevertheless, Eq. (3) remains a good approximation to deceleration.

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References

[1] Stilley, G. D., "Approximate Theory for Terminal Velocity of a Freely Falling Body," *Journal of Spacecraft and Rockets*, Vol. 4, No. 9, Sept. 1967, pp. 1274–1276.

[2] Stengel, R. F., "Wind Profile Measurement Using Sensors," *Journal of Spacecraft and Rockets*, Vol. 3, No. 3, March 1966, pp. 365–373.